MATHEMATICAL EXPRESSION OF GEOLOGIC BOUNDARY BY NEIGHBOURHOOD FUNCTION

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ABSTRACT

The present study has introduced a new concept of geologic units neighbouring around a point P in order to distinguish geologic boundaries included in a 3-D geologic model. We can find a set of geologic units in ε-neighbourhood of a point P by the geologic function. A set of geologic units in ε-neighbourhood of P is expected to be fixed if ε is small enough. The fixed set is called a set of geologic units neighbouring around P. The function which assigns to P a set of geologic units neighbouring around P is called a neighbourhood function. The neighbourhood function provides a mathematical tool to analyze numerically geologic boundaries which composes a 3-D geologic model and leads to an algorithm for extraction of specific boundary points, lines and surfaces from a 3-D geologic model. As an example, it is confirmed that geologic boundary lines can be extracted through a mechanical procedure from a logical model of geologic structure and surfaces given as DEMs.

1. INTRODUCTION

Recently, needs for geologic information have been increasing in various fields such as environmental assessment, urban planning, resource development, waste management and disaster mitigation. It is of urgent necessity to provide geologic information as a three dimensional model for these fields. Shiono *et al*. (1998) have presented *a logical model of geologic structure* as a basis of 3-D geologic modelling. The logical model of geologic structure is a logical relation between geologic units and boundary surfaces. If the logical model of geologic structure defines a partition of 3-D space Ω, we can define a function *g* which assigns a unique geologic unit to every point in the space Ω. This function *g* is called *a geologic function* (Masumoto *et al*., 2004). Based on the geologic function *g*, various geologic modelling systems have been developed, and the geologic function *g* works effectively in these modelling. However, past models used the function *g* only for visualization because the function cannot define explicitly geologic boundary surfaces and lines, which are important information included in 3-D geologic model. It is necessary to develop the theoretical basis for processing geological boundaries to utilize a 3-D geologic model for a design on CAD and a finite element model for various simulations.

In the present study, we propose a new concept of geologic units neighbouring around a point in order to find geologic boundaries included in a 3-D geologic model. We define a neighbourhood function of the geologic function to obtain geologic units neighbouring around a point. Moreover, we show a procedure to extract geologic boundary lines from DEMs by using the neighbourhood function.

2. NEIGHBOURHOOD FUNCTION

2.1 Interior, exterior and boundary in topology

As preparation for defining the neighbourhood function, we review the concept of an interior and boundary in the field of topology. Let Ω be a 3-D metric space where a distance $d(P, P')$ between $P(x, y, z)$ and $P'(x', y', z')$ is defined as follows:

$$d(P, P') = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\ . \tag{1}$$

An open ball about P of radius ε is called a ε-neighbourhood of P, where P is a point in the space Ω and ε is a positive real number. The open ball is denoted by $V(P, \varepsilon)$:

$$V(P, \varepsilon) = \{X |\ X \in \Omega,\ d(X, P) < \varepsilon\}. \tag{2}$$

Let A be a subset of Ω. Then the interior, exterior and boundary of A are defined as follows (Figure 1):

If there is some $\varepsilon > 0$ such that ε-neighbourhood of a point P is included in A, then we say that P is an interior point of A. The set of all interior points is called an interior of A. Let A^c be a complement set of A. Then we also say that an interior point of A^c is an exterior point of A; an interior of A^c is an exterior of A. If there are both interior point and exterior point of A in ε-neighbourhood of P for every $\varepsilon > 0$, then P is a boundary point of A. The set of all boundary points is called a boundary of A. Every point in the space Ω is either interior point or exterior point or boundary point.

2.2 Definition of neighbourhood function

A minimum subspace divided by the boundary surfaces $S_1, S_2, \ldots, S_q$ in a space Ω can be expressed by $m_{d1d2\ldots dq} = h_1(d_1) \cap h_2(d_2) \cap \ldots \cap h_q(d_q)$, where

$$h_i(d_i) = \begin{cases} S_i^+\ ;\ d_i = 1, \\ \\ S_i^-\ ;\ d_i = 0. \end{cases} \tag{3}$$

This minimum subspace $m_{d1d2\ldots dq}$ is called *a minset generated by surfaces* $S_1, S_2, \ldots, S_q$ (Shiono *et al*., 1998). If the surfaces are given, a q-digit binary number $d_1\ d_2 \ldots d_q$ is defined for every point P in the space Ω:

$$d_i = \begin{cases} 0\ ;\ P \text{ lies in the lower side of } S_i \text{ or on } S_i, \\ \\ 1\ ;\ P \text{ lies in the upper side of } S_i. \end{cases} \tag{4}$$

This binary number corresponds to a subscript of minset including a point P. Thus, it is easy to define a function g_2: $\Omega \rightarrow M$ that assigns minset including a point to the point in the space Ω. To formulate an interior point and a boundary point of minset, we introduce a concept of *minsets neighbouring around a point* and define *a neighbourhood function* which assigns minsets neighbouring around a point to the point as follows:

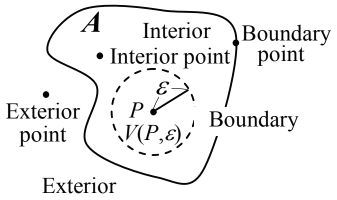

Figure 1. Interior, exterior and boundary of *A*.

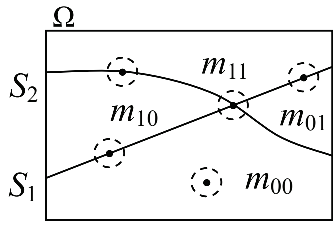

Figure 2. Minsets neighbouring around a point.

Table 1. Logical model of geologic structure.

	S_1	S_2
b_1	0	0
b_2	1	0
b_3	*	1

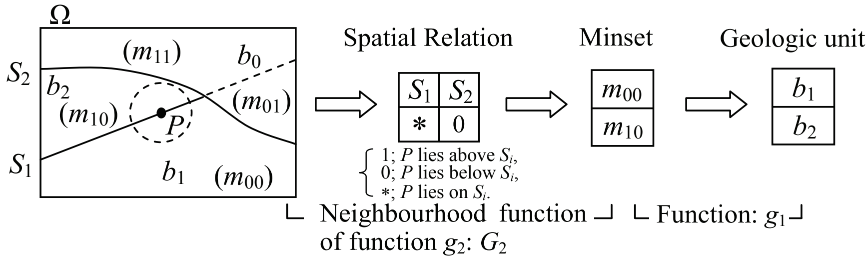

Figure 3. Procedure for generating geologic units neighbouring around a point *P*.

Table 2. Relational code table.

minset	unit
m_{00}	b_1
m_{01}	b_3
m_{10}	b_2
m_{11}	b_3

Let M be a set of minsets generated by surfaces $S_1, S_2, \ldots, S_q$ which divide Ω into two subspaces. Minsets are disjoint to each other. Let $g_2 : \Omega \rightarrow M$ be a function that assigns a minset including a point to the point. Then an image of $V(P, \varepsilon)$ by the function g_2 gives a set of minsets in a ε-neighbourhood of P:

$$g_2(V(P, \varepsilon)) = \{g_2(x)\colon x \in V(P, \varepsilon)\}. \tag{5}$$

If g_2 $(V(P, \varepsilon))$ converges to a certain set of minsets for all ε smaller than a value ε_o, then the set of minsets is denoted by min $g_2(V(P, \varepsilon))$ and the minsets are called *minsets neighbouring around* P (Figure 2). If min $g_2(V(P, \varepsilon))$ exists for all points in the space Ω, we can define a function G_2: $\Omega \rightarrow 2^M$ which assigns min $g_2(V(P, \varepsilon))$ to P. This function G_2 is called *a neighbourhood function of* g_2.

2.3 Neighbourhood function *G* of geologic function *g*

To express interior points and boundary points of geologic units, we introduce a neighbourhood function G of the geologic function g which assigns geologic units neighbouring around a point to the point. Let g_1 be a function from a set of minsets M into a set of geologic units B. Then, the neighbourhood function G of g is a composition of g_1: $M \rightarrow B$ and G_2: G_2: $\Omega \rightarrow 2^M$ (Figure 3):

$$G(P) = g_1(G_2(P)) \,. \tag{6}$$

As an example, steps for finding geologic units neighbouring around the point P in Figure 3 are shown below.

(1) Generate the relational code table from the logical model of geologic structure (Table 1). This table represents the function g_1 (Table 2).

(2) Define a q-digit character code $c_1\, c_2 \ldots c_q$ composed of 0, 1 and * from spatial relations between P and surfaces $S_1, S_2, \ldots, S_q$,

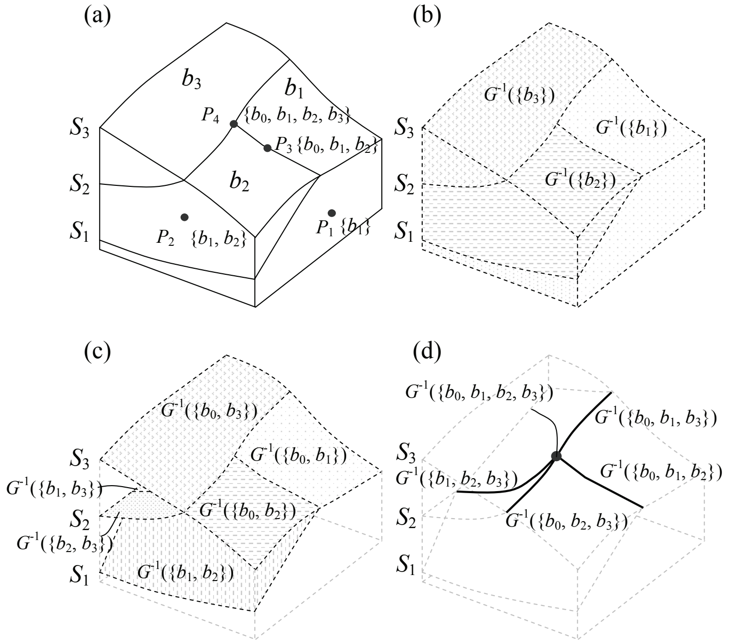

Figure 4. Characteristics of neighbourhood function *G*.

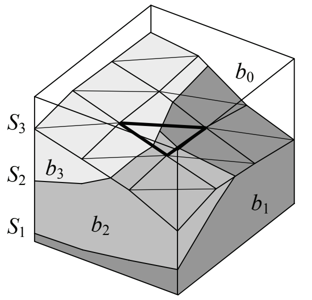

Figure 5. Geologic structure.

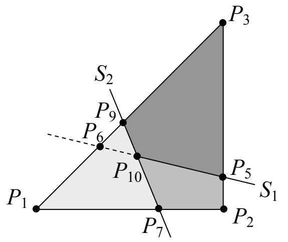

Figure 6. Triangular mesh.

$$c_k = \begin{cases} 0 \text{ ; A point } P \text{ lies below } S_k, \\ 1 \text{ ; A point } P \text{ lies above } S_k, \\ * \text{ ; A point } P \text{ lies on } S_k. \end{cases} \tag{7}$$

In the case of Figure 3, $c_1c_2 = {*}0$.

(3) Generate all q-digit binary numbers $d_1\ d_2 \ldots d_q$ by substituting 0 or 1 for . in the character code $c_1\ c_2 \ldots c_q$. According to this rule, the minsets neighbouring around the point P in Figure 3 are m00 and m10.

(4) $G(P)$ is a set of values of g_1 for minsets specified by q-digit binary numbers $d_1\ d_2 \ldots d_q$ generated in step (3). In the case of this example, the geologic units neighbouring around P are b_1 and b_2.

2.4 Interior and boundary of geologic units by the neighbourhood function *G*

Based on the neighbourhood function G of g, we can express an interior of geologic unit and a boundary of geologic units. Figure 4 gives a simple geologic structure composed of an open space b_0 and three geologic units b_1, b_2 and b_3.

(1) Let $P_1(x_1, y_1, z_1)$ be an interior point of the geologic unit b_1. Then we have $G(x_1, y_1, z_1) = \{b_1\}$. A set of points satisfying $G(x, y, z) = \{b_1\}$, denoted by $G^{-1}(\{b_1\})$, gives an interior of b_1 (see Figure 4(b)).

(2) Let $P_2(x_2, y_2, z_2)$ be a boundary point of b_1 and b_2. Then we have $G(x_2, y_2, z_2) = \{b_1, b_2\}$. A set of points satisfying $G(x, y, z) = \{b_1, b_2\}$, denoted by $G^{-1}(\{b_1, b_2\})$, gives a boundary surface of b_1 and b_2 (see Figure 4(c)).

(3) Let $P_3(x_3, y_3, z_3)$ be a boundary point of b_0, b_1 and b_2. Then we have $G(x_3, y_3, z_3) = \{b_0, b_1, b_2\}$. A set of points satisfying $G(x, y, z) = \{b_0, b_1, b_2\}$, denoted by $G^{-1}(\{b_0, b_1, b_2\})$, gives a boundary line of b_0, b_1 and b_2 (see Figure 4(d)).

(4) Let $P_4(x_4, y_4, z_4)$ be a boundary point of b_0, b_1, b_2 and b_3. Then we have $G(x_4, y_4, z_4) = \{b_0, b_1, b_2, b_3\}$. A set of points satisfying $G(x, y, z) = \{b_0, b_1, b_2, b_3\}$, denoted by $G^{-1}(\{b_0, b_1, b_2, b_3\})$, gives a boundary point of b_0, b_1, b_2 and b_3 (see Figure 4(d)).

3. EXTRACTION OF GEOLOGIC BOUNDARY LINES

Assuming that surfaces S_1, S_2, …, S_q are given as DEMs (Digital Elevation Model), let us consider a method for extracting a boundary line of specified geologic units b_i and b_j on the topographic surface from DEMs. In a computer processing, a curved line is usually approximated by a set of short line segments. In order to draw a boundary line of b_i and b_j on the topographic surface, we need to prepare from DEMs a set of line segments on which b_0, b_i and b_j are adjoining simultaneously. To find an intersection of two surfaces easily, each grid cell of DEM is divided into two triangular meshes. If each triangle is considered as a planar surface, an intersection of two triangles representing different surfaces at the same horizontal position is a line segment which can be specified simply by two points. Repeating the process to find line segments in all triangular meshes, we can define a set of line segments on which b_0, b_i and b_j are adjoining simultaneously.

Figure 5 shows a simple geologic structure which is composed of an open space b_0 and three geologic units b_1, b_2 and b_3. The logical model of this geologic structure is given in Table 3 and the relational code table is given in Table 4. We show practical procedures for extraction of geologic boundary lines from this geologic structure.

We divide each grid cell of DEMs into two triangular meshes and consider the procedures to find a line segment of geologic boundary of b_i and b_j on each triangular mesh of S_3, using the triangular mesh surrounded by a thick line in Figure 5 as an example. Figure 6 shows this triangular mesh.

(1) At first we give sequential numbers to three vertexes of the triangular mesh and all possible intersection points. P_1, P_2 and P_3 are vertexes of a triangular mesh. P_4, P_5 and P_6 are points at which surface S_1 crosses three sides P_1P_2, P_2P_3 and P_1P_3 of the triangle, respectively. P_7, P_8 and P_9 are intersection points of surface S_2 and three sides P_1P_2, P_2P_3 and P_1P_3 of the triangle, respectively. P_{10} is an intersection point of S_1 and S_2 on the triangular mesh.

(2) To find geologic units neighbouring around points P_1, P_2, …, P_{10}, we prepare a table with rows corresponding to points P_1, P_2, …, P_{10} and columns corresponding to five types of attributes shown in Table 5. *Coordinates*: coordinates of P_i will be given. *Triangular sides*: 1 will be given if P_k is located on three sides P_1P_2, P_2P_3 or P_1P_3 of the triangle. *Intersection lines between surface and triangle*: 1 will be given if P_k is located on an intersection line of surface S_k and the triangular mesh. *Spatial relations*: one of 0, 1 and $*$ will be given to show a spatial relation between P_k and each of surfaces S_1, S_2 and S_3. *Neighbouring geologic units*: 1 will be given for geologic units neighbouring around P_k.

(3) We calculate the coordinates of each point and find geologic units neighbouring around the point according to the procedure in the section 2.3. As shown in Table 7, table is completed.

(4) Referring to columns *Neighbouring geologic units*, we can define all line segments of geologic boundaries for specific combinations of geologic units as follows:
   - As $G(P_5) = \{b_0, b_1, b_2\}$ and $G(P_{10}) = \{b_0, b_1, b_2, b_3\}$, P_5P_{10} is a line segment of geologic boundary between b_1 and b_2 on the topographic surface.
   - As $G(P_7) = \{b_0, b_2, b_3\}$ and $G(P_{10}) = \{b_0, b_1, b_2, b_3\}$, P_7P_{10} is a line segment of geologic boundary between b_2 and b_3 on the topographic surface.
   - As $G(P_9) = \{b_0, b_1, b_3\}$ and $G(P_{10}) = \{b_0, b_1, b_2, b_3\}$, P_9P_{10} is a line segment of geologic boundary between b_1 and b_3 on the topographic surface.

Table 3. Logical model of geologic structure.

b_0	*	*	1
	S_1	S_2	S_3
b_1	0	0	0
b_2	1	0	0
b_3	*	1	0

Table 4. Relational code table.

minset	unit
m_{000}	b_1
m_{001}	b_0
m_{010}	b_3
m_{011}	b_0
m_{100}	b_2
m_{101}	b_0
m_{110}	b_3
m_{111}	b_0

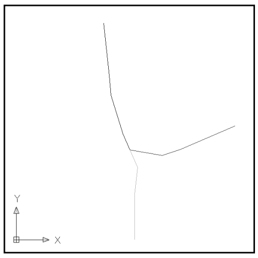

Figure 7. Drawing of line segments.

Table 5. Geologic units neighbouring around a point.

	Coordinates			Triangular sides			Intersection lines between surface and triangle		Spatial relations			Neighboring geologic units			
	x	Y	Z	P_1P_2	P_2P_3	P_1P_3	$S_1 \cap \Delta$	$S_2 \cap \Delta$	S_1	S_2	S_3	b_0	b_1	b_2	b_3
P_1	10.0	10.0	22.0	1		1			1	1	*	1			1
P_2	20.0	10.0	19.0	1	1				1	0	*	1		1	
P_3	20.0	20.0	18.0		1	1			0	0	*	1	1		
P_4	outside														
P_5	20.0	11.7	18.8		1		1		*	0	*	1	1	1	
P_6	12.9	12.9	20.9			1	1		*	1	*	1			1
P_7	16.7	10.0	20.0	1				1	1	*	*	1		1	1
P_8	outside														
P_9	14.6	14.6	20.2			1		1	0	*	*	1	1		1
P_{10}	15.6	12.4	20.1				1	1	*	*	*	1	1	1	1

After repeating the above procedures for all triangular meshes, we can obtain all line segments with attributes specifying geologic units. Figure 7 is an example to draw boundary lines by different colors after converting to DXF format.

4. CONCLUSIONS

Geologic boundaries can be mathematically expressed by the neighbourhood function G of the geologic function g. We have shown the practical procedures for extraction of specific geologic boundary lines from DEMs based on the neighbourhood function G. Though we have extracted only boundary lines as an example, a geologic boundary surface can be expressed as a space surrounded by extracted lines, and a distributed area of the geologic unit can be expressed as a space surrounded by the surfaces. Thus it will be possible to develop some algorithms to define a geologic boundary of specific combination of geologic units and a distributed area of geologic unit from DEMs and to incorporate the algorithm into the 3-D geologic modelling systems.

5. REFERENCES

Masumoto, S., Raghavan, V., Yonezawa, G., Nemoto, T. and Shiono, K., 2004. Construction and visualization of a three dimensional geologic model using GRASS GIS. *Transactions in GIS*, 8(2), 211-223.

Shiono, K., Masumoto, S. and Sakamoto, M., 1994 On Formal Expression of Spatial Distribution of Strata Using Boundary Surface -C1 and C2 Type of Contact-. *Geoinformatics*, 5, 223-232.